Stress intensity factor

Stress linear solution
(Williams, Inglis, Muskhelishvili, Westergaard…)

\[
\sigma_{rr} = \frac{K_I}{4\sqrt{2\pi} r} \left[ 5 \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right]
\]

\[
\sigma_{\theta\theta} = \frac{K_I}{4\sqrt{2\pi} r} \left[ 3 \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \right]
\]

\[
\sigma_{r\theta} = \frac{K_I}{4\sqrt{2\pi} r} \left[ \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right]
\]

\[
[u] = u|_{x^+} - u|_{x^-} = \frac{4(1 - \nu)}{\mu} \sqrt{\frac{r}{2\pi}} K_I e_y
\]

\[
\sigma_{ij} = O \left( \frac{1}{\sqrt{r}} \right)
\]
Stress intensity factor
George Rankine Irwin (1907-1998)

Stress intensity factors are formally defined by

\[
\begin{align*}
\lim_{r \to 0} \sigma_{ij}^{(I)} &= \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{(I)}(\theta) \\
\lim_{r \to 0} \sigma_{ij}^{(II)} &= \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{(II)}(\theta) \\
\lim_{r \to 0} \sigma_{ij}^{(III)} &= \frac{K_{III}}{\sqrt{2\pi r}} f_{ij}^{(III)}(\theta)
\end{align*}
\]

For mixed modes:

\[
\sigma_{ij} = \sigma_{ij}^{(I)} + \sigma_{ij}^{(II)} + \sigma_{ij}^{(III)}
\]

They depend on sample geometry, the size and location of the crack, and the magnitude and the modal distribution of loads on the material.
Stress intensity factor

When the loading mode is defined, all effects related to load, geometry, and crack dimensions, are accounted for by the stress intensity factor $K$.

\[
\sigma_{rr} = \frac{K_I}{4\sqrt{2}\pi r} \left[ 5 \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right]
\]

\[
\sigma_{\theta\theta} = \frac{K_I}{4\sqrt{2}\pi r} \left[ 3 \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \right]
\]

\[
\sigma_{r\theta} = \frac{K_I}{4\sqrt{2}\pi r} \left[ \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right]
\]

\[
\sigma_{yy}(r, 0) = \sigma_{\theta\theta}(r, 0) \propto \frac{K_I}{\sqrt{2} \pi r}
\]

\[
K_I = \lim_{r \to 0} \left[ \sqrt{2\pi r} \sigma_{yy}(r, 0) \right]
\]

Since the linearity, the local stress has to be proportional to the remote uniform stress state (namely, $\sigma_{yy}^\infty$). Moreover, the local stress has to be affected by the crack dimension $a$. Since the occurrence of $\sqrt{r}$ at denominator of $\sigma_{yy}$, for dimensional reasons the dependency of $\sigma_{yy}$ on $a$ has to be via $\sqrt{a}$. Accordingly, it results

\[
\sigma_{yy} \propto \frac{\sigma_{yy}^\infty \sqrt{a}}{\sqrt{2\pi r}} \propto \frac{\sigma_{yy}^\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \Rightarrow K_I \propto \sigma_{yy}^\infty \sqrt{\pi a}
\]
Examples of stress intensity factors

- Crack in an infinite plate under mode I loading
- Penny-shaped crack of radius \( a \) in an infinite domain under uniaxial tension

\[
K_I = \sigma \sqrt{\pi a}
\]

\[
K_I = \frac{2}{\pi} \sigma \sqrt{\pi a}
\]

\[
\cdot d = b \quad K_I = \sigma \sqrt{\pi a} \left[ \frac{1 - \frac{a}{2b} + 0.326 \left( \frac{a}{b} \right)^2}{\sqrt{1 - \frac{a^2}{b^2}}} \right]
\]

\[
\cdot d \neq b \quad K_{IA} = \sigma \sqrt{\pi a} \Phi_A, \quad K_{IB} = \sigma \sqrt{\pi a} \Phi_B
\]

\[
\Phi_A = \left[ \beta + \left( \frac{1 - \beta}{4} \right) \left( 1 + \frac{1}{4 \sec \alpha_A} \right)^2 \right] \sqrt{\sec \alpha_A}
\]

\[
\Phi_B = 1 + \frac{\sqrt{\sec \alpha_{AB} - 1}}{1 + 0.21 \sin \left\{ 8 \tan^{-1} \left[ \frac{\alpha_A - \alpha_B}{\alpha_A + \alpha_B} \right] \right\}}
\]

\[
\beta = \sin \left( \frac{\pi \alpha_B}{\alpha_A + \alpha_B} \right), \quad \alpha_A = \frac{\pi a}{2d}, \quad \alpha_B = \frac{\pi a}{4b - 2d};
\]

\[
\alpha_{AB} = \frac{4}{3} \alpha_A + \frac{3}{7} \alpha_B
\]
Examples of stress intensity factors

- $h/b \geq 1$, $a/b \leq 0.6$
  \[ K_I = \sigma \sqrt{\pi a} \left[ 1.12 - 0.23 \left( \frac{a}{b} \right) + 10.6 \left( \frac{a}{b} \right)^2 - 21.7 \left( \frac{a}{b} \right)^3 + 30.4 \left( \frac{a}{b} \right)^4 \right] \]

- $h/b \geq 1$, $a/b \geq 0.3$
  \[ K_I = \sigma \sqrt{\pi a} \left[ \frac{1 + 3 \frac{a}{b}}{2 \sqrt{\pi a} \left( 1 - \frac{a}{b} \right)^{3/2}} \right] \]

For a slanted crack in a thin plate under biaxial load:

- $K_I = \sigma \sqrt{\pi a} \left( \cos^2 \beta + \alpha \sin^2 \beta \right)$
- $K_{II} = \sigma \sqrt{\pi a} \left( 1 - \alpha \right) \sin \beta \cos \beta$

\[ K \propto \sigma \sqrt{\pi a} \]
Examples of stress intensity factors

In general,

\[ K = \beta \sigma \infty \sqrt{\pi a} \]

with \( \beta \) a positive dimensionless coefficient depending on the aspect ratios of the structural element wherein the crack is placed.

Remark. As a convention, previous relationships hold if cracks with two apexes are defined as \( 2a \) long, and cracks with only one apex are \( a \) long.

\[ \beta = 1 \]

\[ \beta = \left[ \frac{4b}{\pi a} \tan\left(\frac{\pi a}{4b}\right) \right]^{1/2} \]

\[ \beta = \sqrt{\sec \frac{\pi a}{b}} \]
Irwin approach

A crack propagation criterion only based on stress levels near the crack tip is not realiable!!!!

Irwin’s idea

propagation criterion not expressed in terms of stress state but in terms of the stress intensity factor

\[ K_I < K_{Ic} \quad \text{no crack propagation} \]
\[ K_I = K_{Ic} \quad \text{crack propagation} \]

\[ K_{Ic} = \text{critical stress intensity factor (thoughness)} \]

For a fixed loading mode, \( K_{Ic} \) is experimentally proven to be almost independent on geometry and loading conditions. Thereby, it can be considered as a material property. For steel it is in the order of \( 50 \div 150 \text{ MPa}\sqrt{\text{m}} \)
Irwin approach

Crack propagation (fracture) $\iff K = K_c$

$\Rightarrow \beta \sigma^\infty \sqrt{\pi a} = K_c \quad \Rightarrow \quad \sigma^\infty_{fr} = \frac{K_c}{\beta \sqrt{\pi a}}$

$\sigma^\infty_{fr}$ : nominal fracture stress
Plastic region at the crack tip

\[
\sigma_{rr} = \frac{K_I}{4\sqrt{2\pi r}} \left[ 5 \cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \right] + \frac{K_{II}}{4\sqrt{2\pi r}} \left[ -5 \sin\frac{\theta}{2} + 3 \sin\frac{3\theta}{2} \right]
\]

\[
\sigma_{\theta\theta} = \frac{K_I}{4\sqrt{2\pi r}} \left[ 3 \cos\frac{\theta}{2} + \cos\frac{3\theta}{2} \right] + \frac{K_{II}}{4\sqrt{2\pi r}} \left[ -3 \sin\frac{\theta}{2} - 3 \sin\frac{3\theta}{2} \right]
\]

\[
\sigma_{r\theta} = \frac{K_I}{4\sqrt{2\pi r}} \left[ \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \right] + \frac{K_{II}}{4\sqrt{2\pi r}} \left[ \cos\frac{\theta}{2} + 3 \cos\frac{3\theta}{2} \right]
\]

- Mode I: \( \sigma^\infty = \sigma^\infty_{yy} e_y \otimes e_y \Rightarrow K_I \neq 0, K_{II} = 0 \)
- Mode II: \( \sigma^\infty = \sigma^\infty_{xy} e_x \otimes e_y \Rightarrow K_I = 0, K_{II} \neq 0 \)
- Mixed mode I+II: \( \sigma^\infty = \sigma^\infty_{yy} e_y \otimes e_y + \sigma^\infty_{xy} e_x \otimes e_y \Rightarrow K_I \neq 0, K_{II} \neq 0 \)

\[
\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[ 1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right] - \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \left[ 2 \cos\frac{\theta}{2} \cos\frac{3\theta}{2} \right]
\]

\[
\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[ 1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right] + \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\frac{3\theta}{2}
\]

\[
\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cos\frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[ 1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right]
\]

\[\sigma_{zz} = 0 \text{ for plane stress;} \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \text{ for plane strain}\]

\[\sigma_{xz} = \sigma_{yz} = 0\]
Plastic region at the crack tip

Referring to the mode I, for \( \theta = 0 \):

\[
\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}, \quad \sigma_{xy} = 0
\]

By adopting Tresca criterion:

for plane strain \( \sigma_{eq} = \sigma_{yy}(1 - 2\nu) \)

for plane stress \( \sigma_{eq} = \sigma_{yy} \)

Referring to an ideal plastic behaviour, since the yielding stress \( \sigma_o \)
can not be overcome, then in the region \( r^* \) long the condition \( \sigma_{eq} = \sigma_o \) holds.

As a result, for \( r \in (0, r^*] \) the local stresses \( \sigma_{xx} = \sigma_{yy} \) attain the value:

\[
\sigma^* = \frac{\sigma_o}{1 - 2\nu} \quad \text{(plane strain)}, \quad \sigma^* = \sigma_o \quad \text{(plane stress)}
\]

and \( r^* \) results from

\[
\frac{K}{\sqrt{2\pi r^*}} = \sigma^* \quad \Rightarrow \quad r^* = \frac{K^2}{2\pi(\sigma^*)^2}
\]
Plastic region at the crack tip

It is worth observing that plastic zone is greater than $r^*$, since redistribution effects of local stresses. In detail, the stress part exceeding the value $\sigma^*$ corresponds to a loading part that, for equilibrium, has to be carried through additional stresses out of the region $r^*$ long. As a result, the zone where plasticity is reached is characterized by a length scale $r_p > r^*$. In the case of plane stress ($\sigma^* = \sigma_o$), vertical equilibrium on the line for $\theta = 0$, leads to:

$$\sigma_o r_p = \int_0^{r^*} \frac{K}{\sqrt{2\pi r}} \, dr = \frac{2K \sqrt{r^*}}{\sqrt{2\pi}}$$

Under the same nominal stress, a plane strain condition is associated to a smaller plastic region than the case of plane stress. Accordingly, the dissipation level is smaller and the elastic energy that could be relaxed in a brittle fracture is greater. As a result, the plane strain condition is more dangerous than the plane stress one.
Plastic zone shape at the crack tip

Let the Von Mises criterion be considered, so that the equivalent stress is:
\[
\sigma_{eq} = \frac{1}{\sqrt{2}} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_3 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 \right]^{1/2}
\]

where, referring to the mode I, the principal stresses \(\sigma_i\) (with \(i = 1, 2, 3\)) are determined by the stress solution:
\[
\begin{align*}
\sigma_{1,2} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left( \frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 \pm \sin \frac{\theta}{2} \right) \\
\sigma_3 &= \begin{cases} 
0 & \text{plane stress} \\
\nu(\sigma_1 + \sigma_2) = \frac{2\nu K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} & \text{plane strain}
\end{cases}
\end{align*}
\]

By enforcing \(\sigma_{eq} = \sigma_o\) and by solving with respect to \(r\), one obtains the radius of the plastic zone as a function of \(\theta\):
- plane stress \(r(\theta) = \frac{1}{4\pi} \frac{K_I^2}{\sigma_o^2} \left( 1 + \cos \theta + \frac{3}{2} \sin^2 \theta \right)\)
- plane strain \(r(\theta) = \frac{1}{4\pi} \frac{K_I^2}{\sigma_o^2} \left[ (1 - 2\nu)^2(1 + \cos \theta) + \frac{3}{2} \sin^2 \theta \right]\)
Plastic zone shape at the crack tip

Effect of thickness on plastic zone shape (Von Mises-based)

Plastic zone shape (Von Mises-based)
Plasticity effects in linearly-elastic fracture

Since stresses have to be finite values at the crack tip, a plastic zone occurs. If $r_p$ is small, then the stress intensity factor determined via a linearly-elastic solution can be considered as accurate. On the other hand, if $r_p$ is not negligible (namely, if it has the same order of magnitude of $a$), then the linear approach is no longer applicable. In these latter cases, if $r_p$ is not too much large, corrections with respect to the linear approach can be adopted.

Irwin's correction:

$$a_{eff} = a + r^*$$

with $r^* = K^2/[2\pi(\sigma^*)^2]$ and $\sigma^* = \sigma_0$ in plane stress and $\sigma^* = \sigma_0/(1 - 2\nu)$ in plane strain. Accordingly, an effective stress intensity factor is introduced as:
Plasticity effects in linearly-elastic fracture

Irwin’s correction:

\[ a_{eff} = a + r^* \Rightarrow K_{eff} = \beta(a_{eff})\sqrt{\pi a_{eff}} \]

If (as it is usual) \( \beta \) explicitly depends on \( a \), then the determination of \( K_{eff} \) requires an iterative approach:

1. determine an estimate for \( K_{eff} \) by considering \( r^* = 0 \) (i.e., \( K_{eff} = K \));
2. compute an estimate for \( r^* \) as \( r^* = K_{eff}^2/[2\pi(\sigma^*)^2] \)
   by employing the previous estimate for \( K_{eff} \);
3. give a new estimate for \( K_{eff} \) by adopting \( a_{eff} = a + r^* \);
4. repeat from the point 2 until the convergence of the value of \( K_{eff} \).

In cases when \( \beta \) does not depend on \( a \) then it is possible to find a closed-form relationship for \( K_{eff} \). This is the case of a crack in an infinite plate under mode I loading (\( K = \sigma\sqrt{\pi a}, \beta = 1 \)). Addressing this case under the plane stress assumption (\( \sigma^* = \sigma_o \)):

\[ K_{eff} = \frac{\sigma\sqrt{\pi a}}{\sqrt{1 - \frac{1}{2}\left(\frac{\sigma}{\sigma_o}\right)^2}} \]