Original Griffith theory (1920)

Assumptions

- Linearly-elastic material, homogeneous medium
- External loads: only constant surface tractions $p$ on $\partial \Omega_p$
- Quasi-static reversible thermodynamic process

\[ \partial \Omega = \partial \Omega_u \cup \partial \Omega_p \]

$\mathcal{A}$: crack surface area

\[ u = U = \text{const} \]

Total energy system

\[ \mathcal{E} = W - \Phi + \mathcal{U}_s \]

- Elastic strain energy
  \[ W = \frac{1}{2} \int_{\Omega} \mathbb{C} \hat{\nabla} u : \hat{\nabla} u \, d\Omega \quad \text{with} \quad \hat{\nabla} u = \text{sym}(\nabla u) \]

- External loads work
  \[ \Phi = \int_{\partial \Omega_p} p \cdot u \, dS + \int_{\partial \Omega_u} \sigma n \cdot U \, dS \]

- Potential energy

- Surface energy
  \[ \gamma \cdot \text{adhesion surface energy per area unit} \]

Onset crack-surface growth $\Leftrightarrow \mathcal{E}$’s stationarity $\Leftrightarrow$ system’s equilibrium

\[ \frac{d\mathcal{E}}{dA} = 0 \quad \Leftrightarrow \quad -\frac{d}{dA} (W - \Phi) = \frac{d\mathcal{U}_s}{dA} \]
Original Griffith theory (1920)

\[ \Omega \]

\[ \partial \Omega = \partial \Omega_u \cup \partial \Omega_p \]

\[ u = U = \text{const} \]

\[ A^+ \]

\[ A^- \]

\[ A \]

\[ \mathbf{n} \]

\[ \mathbf{p} \]

Onset crack-surface growth \( \Leftrightarrow \mathcal{E}' \)’s stationarity \( \Leftrightarrow \) system’s equilibrium

\[ \frac{d\mathcal{E}}{dA} = 0 \quad \Leftrightarrow \quad -\frac{d}{dA} (W - \Phi) = \frac{dU_s}{dA} \]

\[ G = -\frac{d}{dA} (W - \Phi) = -\frac{dP}{dA} \]

energy release rate

\[ G_c = \frac{dU_s}{dA} = 2\gamma \]

critical energy release rate

Griffith’s criterion

\[ d\Phi = dW + 2\gamma dA \]

At the onset of the crack-surface growth, the work increment of external loads (at constant loads) contributes on one hand to the increment of the elastic strain energy and on the other hand to produce an increment of the surface energy, as the result of the crack extension.

Although time variable did not appeared in the quasi-static Griffith’s approach, Griffith’s criterion can be recast in terms of a power balance:

\[ \dot{\Phi} = \dot{W} + 2\gamma \dot{A} \]

\[ (G - 2\gamma) \dot{A} = 0 \]

External power \( \dot{\Phi} = \int_{\partial \Omega_p} \mathbf{p} \cdot \dot{\mathbf{u}} dS \) since \( \dot{U} = 0 \) on \( \partial \Omega_u \)

Internal power \( \dot{W} = \int \mathbf{C} \nabla \mathbf{u} : \nabla \dot{\mathbf{u}} d\Omega \)

Surface power \( \dot{U}_s = 2\gamma \dot{A} \)
Limitations of the Griffith theory

- Griffith’s power balance implicitly corresponds to the first principle of thermodynamics. But where is heat?

- The condition \((G - 2\gamma)\dot{A} = 0\) has been derived under the only condition \(\ddot{A} \neq 0\), without any request on the sign of \(\dot{A}\). Accordingly, Griffith approach implies the process reversibility. This is non-physical. As a matter of fact, term \(2\gamma\) should be considered as an irreversible dissipated energy when crack propagates.

- Griffith’s theory considers surface tractions only and boundary displacements, both as constant in time.
Overcoming limitations of the Griffith theory

\[ \mathcal{F}(t) = c_f(t)\mathcal{F}_o \]

Assumptions
- Linearly-elastic material, homogeneous medium
- Time-variable external loads and boundary displacements
- Irreversible isothermal process
- Reversible surface energy is negligible. Crack growth induces a constant dissipation rate \((G_c \text{ per crack area unit})\)

loading factors
\[ c(t) = (c_u(t), c_p(t), c_f(t)) \]

\[ D = \mathcal{P} - \dot{W} \geq 0 \]

Clausius-Duhem inequality

\[ \mathcal{P} = \int_{\Omega} \rho \mathcal{F} \cdot \dot{\mathbf{u}} \, d\Omega + \int_{\partial \Omega_p} \mathbf{p} \cdot \dot{\mathbf{u}} \, dS + \int_{\partial \Omega_u} \mathbf{\sigma} \cdot \dot{\mathbf{u}} \, dS \]

\[ = c_f(t) \int_{\Omega} \rho \mathcal{F}_o \cdot \dot{\mathbf{u}} \, d\Omega + c_p(t) \int_{\partial \Omega_p} \mathbf{p}_o \cdot \dot{\mathbf{u}} \, dS + c_u(t) \int_{\partial \Omega_u} \mathbf{\sigma} \cdot \mathbf{u}_o \, dS \]

\[ \dot{W} = \int_{\Omega} \mathbb{C} \mathbf{\nabla u} : \mathbf{\nabla} \dot{\mathbf{u}} \, d\Omega \]

\( D \): dissipation
\( \mathcal{P} \): external power (associated to the external forces and reactions)
\( \dot{W} \): internal power (associated to the elastic strain)
Overcoming limitations of the Griffith theory

At time $t$, displacement field depends on external loads and on the actual configuration. Namely, $u = u(c(t), A(t))$

$$\dot{u} = \frac{\partial u}{\partial c} \dot{c} + \frac{\partial u}{\partial A} \dot{A}$$

$$P = P|_A + P|_e$$

$$P|_A = \left[ c_f(t) \int_{\Omega} \rho F_o \cdot \frac{\partial u}{\partial c} \, d\Omega + c_p(t) \int_{\partial \Omega_p} p_o \cdot \frac{\partial u}{\partial c} \, dS \right] \dot{c} + c_u(t) \int_{\partial \Omega_u} \sigma n \cdot u_o \, dS$$

$$P|_e = \left[ c_f(t) \int_{\Omega} \rho F_o \cdot \frac{\partial u}{\partial A} \, d\Omega + c_p(t) \int_{\partial \Omega_p} p_o \cdot \frac{\partial u}{\partial A} \, dS \right] \dot{A}$$

$$\dot{W} = \int_{\Omega} C \nabla u : \nabla \dot{u} \bigg|_A d\Omega \dot{c} + \int_{\Omega} C \nabla u : \nabla \dot{u} \bigg|_e d\Omega \dot{A} = \frac{\partial W}{\partial c} \bigg|_A \dot{c} + \frac{\partial W}{\partial A} \bigg|_e \dot{A}$$

$$P|_c = \frac{\partial \Phi}{\partial A} \bigg|_c \dot{A}$$

Work of external forces

$$\Phi = \int_{\Omega} \rho F \cdot u \, d\Omega + \int_{\partial \Omega_p} p \cdot u \, dS = c_f(t) \int_{\Omega} \rho F_o \cdot u \, d\Omega + c_p(t) \int_{\partial \Omega_p} p_o \cdot u \, dS$$

$$P|_A = \frac{\partial W}{\partial c} \bigg|_A \dot{c}$$

since the Principle of the Virtual Powers
Overcoming limitations of the Griffith theory

\[ \mathcal{P}[A] = \frac{\partial W}{\partial c} \bigg|_A \dot{c} \]

since the Principle of the Virtual Powers

\[ \mathcal{P}[A] = \left[ c_f(t) \int_{\Omega} \rho \mathbf{F}_o \cdot \frac{\partial \mathbf{u}}{\partial c} d\Omega + c_p(t) \int_{\partial \Omega_p} p_o \cdot \frac{\partial \mathbf{u}}{\partial c} dS \right] \dot{c} + \dot{c}_u(t) \int_{\partial \Omega_u} \sigma \mathbf{n} \cdot \mathbf{u}_o dS \]

\[ \dot{W} = \int_{\Omega} \nabla \mathbf{u} : \hat{\nabla} \frac{\partial \mathbf{u}}{\partial c} \bigg|_A d\Omega \dot{c} + \int_{\Omega} \nabla \mathbf{u} : \hat{\nabla} \frac{\partial \mathbf{A}}{\partial c} \bigg|_A d\Omega \dot{A} = \frac{\partial W}{\partial c} \bigg|_A \dot{c} + \frac{\partial W}{\partial \dot{A}} \bigg|_A \dot{A} \]

\[ \frac{\partial W}{\partial c} \bigg|_A \dot{c} = \int_{\Omega} \nabla \mathbf{u} : \hat{\nabla} \frac{\partial \mathbf{u}}{\partial c} \bigg|_A d\Omega \dot{c} \]

\[ \mathcal{P}[A] \delta t = \left[ c_f(t) \int_{\Omega} \rho \mathbf{F}_o \cdot \frac{\partial \mathbf{u}}{\partial c} d\Omega + c_p(t) \int_{\partial \Omega_p} p_o \cdot \frac{\partial \mathbf{u}}{\partial c} dS \right] \frac{dc}{dt} \delta t + \frac{dc_u(t)}{dt} \delta t \int_{\partial \Omega_u} \sigma \mathbf{n} \cdot \mathbf{u}_o dS \]

\[ = \int_{\Omega} \rho \mathbf{F} \cdot \delta \mathbf{u} d\Omega + \int_{\partial \Omega_p} p \cdot \delta \mathbf{u} dS + \int_{\partial \Omega_u} \sigma \mathbf{n} \cdot \delta \mathbf{u} dS = \delta L_e \]

\[ \delta \mathbf{u} = \delta c_u \mathbf{u}_o \quad \text{on } \partial \Omega_u \]

\[ \frac{\partial W}{\partial c} \bigg|_A \dot{c} \delta t = \int_{\Omega} \nabla \mathbf{u} : \hat{\nabla} \frac{\partial \mathbf{u}}{\partial c} \bigg|_A d\Omega \frac{dc}{dt} \delta t = \int_{\Omega} \nabla \mathbf{u} : \hat{\nabla} \delta \mathbf{u} \bigg|_A d\Omega = \delta L_i \]
Overcoming limitations of the Griffith theory

\[ D = \mathcal{P} - \dot{W} \geq 0 \Rightarrow (\mathcal{P}|_A + \mathcal{P}|_c) - \left( \frac{\partial W}{\partial c}_A \dot{c} + \frac{\partial W}{\partial A}_c \dot{A} \right) = \frac{\partial \Phi}{\partial A}_c \dot{A} - \frac{\partial W}{\partial A}_c \dot{A} \geq 0 \]

\[ \mathcal{P}|_c = \frac{\partial \Phi}{\partial A}_c \dot{A} \]

\[ \mathcal{P}|_A = \frac{\partial W}{\partial c}_A \dot{c} \]

\[ D = -\frac{\partial}{\partial A} (W - \Phi)|_c \dot{A} \geq 0 \]

\[ G(c, A) = -\frac{\partial}{\partial A} (W - \Phi)|_c \Rightarrow D = G(c, A) \dot{A} \geq 0 \]

• Dissipation does not depend on the work of external forces for a fixed configuration (namely, when crack does not propagate). Dissipation is fully related to crack growth and dissipation tends to zero when \( \dot{A} \rightarrow 0 \).

• The energy release rate \( G(c, A) \) clearly assumes the meaning of thermodynamic driving fracture force, resulting the dual quantity to \( \dot{A} \) in the Clausius-Duhem inequality.
Overcoming limitations of the Griffith theory

Crack propagation criterion

\[ G(c, A) < G_c \implies \dot{A} = 0 \]
\[ G(c, A) = G_c \implies \dot{A} \geq 0 \]

- \( G_c \) is a material property (thoughness). It is constant for perfectly brittle materials. In quasi-brittle or ductile materials \( G_c \) generally varies with \( A \).

- The original Griffith’s criterion is recovered, by eliminating the physical inconsistency related to the possibility to have also \( \dot{A} < 0 \) (i.e., reversibility).

- Let a crack propagation state be considered under the assumption \( G_c = \text{const} \), that is let \( c \) and \( A \) such that \( G(c, A) = G_c \) (thereby, \( \dot{A} \geq 0 \)). Starting form this state, let a crack perturbation \( dA > 0 \). If \( G(c, A + dA) < G(c, A) = G_c \) then crack propagation stops. Thereby, the state \( (c, A) \) identifies a stable propagation state. If \( G(c, A + dA) > G(c, A) = G_c \) then crack propagation occurs under constant loads. Thereby, the state \( (c, A) \) identifies an unstable propagation state. Accordingly

\[
\lim_{dA \to 0} \frac{G(c, A + dA) - G(c, A)}{dA} = -\frac{\partial^2}{\partial A^2} (W - \Phi)|_c > 0 \quad \text{Unstable crack propagation}
\]
Relationship between G and K: Irwin theory

\[ \partial \Omega_o = \partial \Omega_p \cup \partial \Omega_u \]
\[ \partial \Omega(A) = \partial \Omega_o \cup A \]
\[ A = A^+ \cup A^- \]

\[ t = \sigma n = 0 \quad \text{on } A^+ \cup A^- \]

- Strain energy (due to Clapeyron)
  \[ W = \frac{1}{2} \left( \int_{\partial \Omega_o} t \cdot u \, dS + \int_{\Omega(A)} \rho \mathbf{F} \cdot u \, d\Omega \right) \]
  with \( t = \sigma n \) on \( \partial \Omega_u \) and \( t = p \) on \( \partial \Omega_p \)

- Work of external forces (only!)
  \[ \Phi = \int_{\partial \Omega_p} p \cdot u \, dS + \int_{\Omega(A)} \rho \mathbf{F} \cdot u \, d\Omega \]

- Potential energy:
  \[ P = W - \Phi = \frac{1}{2} \left( \int_{\partial \Omega_u} t \cdot U \, dS - \int_{\partial \Omega_p} p \cdot u \, dS - \int_{\Omega(A)} \rho \mathbf{F} \cdot u \, d\Omega \right) \]
Irwin theory

Energy release rate:

\[ G(c, A) = -\frac{\partial P}{\partial A}\bigg|_c \]

\((\cdot)|_c\) means at generalized loads (namely, \(\mathcal{F}, p, U\)) considered as constant.

\[ P = W - \Phi = \frac{1}{2} \left( \int_{\partial \Omega_u} \mathbf{t} \cdot \mathbf{U} \, dS - \int_{\partial \Omega_p} \mathbf{p} \cdot \mathbf{u} \, dS - \int_{\Omega(A)} \rho \mathcal{F} \cdot \mathbf{u} \, d\Omega \right) \]

\[ G(c, A) = -\frac{1}{2} \left( \int_{\partial \Omega_u} \frac{\partial \mathbf{t}}{\partial A} \cdot \mathbf{U} \, dS - \int_{\partial \Omega_p} \mathbf{p} \cdot \frac{\partial \mathbf{u}}{\partial A} \, dS - \int_{\Omega(A)} \rho \mathcal{F} \cdot \frac{\partial \mathbf{u}}{\partial A} \, d\Omega \right) \]

\[ = \frac{1}{2} \left[ \int_{\partial \Omega_o} \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial A} - \frac{\partial \mathbf{t}}{\partial A} \cdot \mathbf{u} \right] \, dS + \int_{\Omega(A)} \rho \mathcal{F} \cdot \frac{\partial \mathbf{u}}{\partial A} \, d\Omega \]

since \[ \int_{\partial \Omega_u} \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial A} \, dS = \int_{\partial \Omega_p} \frac{\partial \mathbf{t}}{\partial A} \cdot \mathbf{u} \, dS = 0. \]
Irwin theory

\[ G(c, A) = \frac{1}{2} \left[ \int_{\partial \Omega_o} \left( \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial A} - \frac{\partial \mathbf{t}}{\partial A} \cdot \mathbf{u} \right) dS + \int_{\Omega(A)} \rho \mathbf{F} \cdot \frac{\partial \mathbf{u}}{\partial A} d\Omega \right] \]

\[ 2G(c, A) \, dA = \int_{\partial \Omega_o} \mathbf{t}_1 \cdot (\mathbf{u}_2 - \mathbf{u}_1) + \mathbf{u}_1 \cdot (\mathbf{t}_1 - \mathbf{t}_2) dS + \int_{\Omega(A)} \rho \mathbf{F} \cdot (\mathbf{u}_2 - \mathbf{u}_1) d\Omega = D_{12} - D_{21} \]

with

\[ D_{12} = \int_{\partial \Omega_o} \mathbf{t}_1 \cdot \mathbf{u}_2 dS + \int_{\Omega(A)} \rho \mathbf{F} \cdot \mathbf{u}_2 d\Omega \]

\[ D_{21} = \int_{\partial \Omega_o} \mathbf{t}_2 \cdot \mathbf{u}_1 dS + \int_{\Omega(A)} \rho \mathbf{F} \cdot \mathbf{u}_1 d\Omega \]

\[ \int_{\Omega(A)} (\cdot) d\Omega = \int_{\Omega(A+dA)} (\cdot) d\Omega \]
Irwin theory

\[ D_{21} = \int_{\partial \Omega_o} \mathbf{t}_2 \cdot \mathbf{u}_1 dS + \int_{\Omega(A+dA)} \rho \mathbf{F} \cdot \mathbf{u}_1 d\Omega = \int_{\Omega(A+dA)} \mathbf{\sigma}_2 : \mathbf{\epsilon}_1 d\Omega + \int_{dA^+} \mathbf{t}_2 \cdot \mathbf{u}_1 dS + \int_{dA^-} \mathbf{t}_2 \cdot \mathbf{u}_1 dS \]

\[ = \int_{\Omega(A+dA)} \mathbf{\sigma}_2 : \mathbf{\epsilon}_1 d\Omega \]

\[ \mathbf{t}_2 = 0 \text{ on } dA^+ \cup dA^- \]

\[ D_{12} = \int_{\partial \Omega_o} \mathbf{t}_1 \cdot \mathbf{u}_2 dS + \int_{\Omega(A+dA)} \rho \mathbf{F} \cdot \mathbf{u}_2 d\Omega = \int_{\Omega(A+dA)} \mathbf{\sigma}_1 : \mathbf{\epsilon}_2 d\Omega + \int_{dA^+} \mathbf{t}_1 \cdot \mathbf{u}_2 dS + \int_{dA^-} \mathbf{t}_1 \cdot \mathbf{u}_2 dS \]

\[ = \int_{\Omega(A+dA)} \mathbf{\sigma}_1 : \mathbf{\epsilon}_2 d\Omega + \int_{\partial \Omega} \mathbf{t}_1 \cdot [\mathbf{u}_2] dS \]
Irwin theory

\[ D_{21} = \int_{\Omega(A+dA)} \sigma_2 : \varepsilon_1 \, d\Omega \]
\[ D_{12} = \int_{\Omega(A+dA)} \sigma_1 : \varepsilon_2 \, d\Omega + \int_{dA} t_1 \cdot [u_2] \, dS \]

By applying the Betti’s theorem:
\[ \int_{\Omega(A+dA)} \sigma_1 : \varepsilon_2 \, d\Omega = \int_{\Omega(A+dA)} \sigma_2 : \varepsilon_1 \, d\Omega \]

\[ \Rightarrow \quad D_{12} - D_{21} = \int_{dA} t_1 \cdot [u_2] \, dS \quad \Rightarrow \quad 2G(c, A)dA = D_{12} - D_{21} = \int_{dA} t_1 \cdot [u_2] \, dS \]
Irwin theory

\[ 2G(c, A) dA = \int_{dA} t_1 \cdot [u_2] \, dS \]

Plane-state assumption

\[ dA = L \, d\ell \quad \Rightarrow \quad \int_{dA} (\cdot) \, dS = L \int_0^{d\ell} (\cdot) \, dx \]

\[ t_1|_{dA^+} = \sigma_1|_{(\theta=0,r)} n^+ = \sigma_1|_{(\theta=0,r)} (-j) \]

with \((\theta = 0)\) \( r \equiv x \in [0, d\ell] \)

\[ [[u_2]]|_{dA} = [[u_2(\theta = \pi, r)]] = [[u_2(\pi, d\ell - x)]] \]

with \((\theta = \pi)\) \( r \equiv -x' = (d\ell - x) \in [0, d\ell] \)

\[ 2G(c, \ell) \, d\ell = \int_0^{d\ell} t_1 \cdot [u_2] \, dx \]
**Irwin theory**

**Williams solution**

\[
\sigma_{rr} = \frac{K_I}{4\sqrt{2\pi r}} \left[ 5 \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right] \\
\sigma_{\theta\theta} = \frac{K_I}{4\sqrt{2\pi r}} \left[ 3 \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \right] \\
\sigma_{r\theta} = \frac{K_I}{4\sqrt{2\pi r}} \left[ \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right]
\]

\[
[u] = u|_{x^+} - u|_{x^-} = \frac{4(1-\nu)}{\mu} \sqrt{\frac{r}{2\pi}} K_I e_y
\]

\[
2G(c, \ell) d\ell = \int_0^{d\ell} \mathbf{t}_1 \cdot [[u_2]] dx
\]

**Irwin equation**

\[
G(c, \ell) = \frac{1 - \nu^2}{E} K_I^2
\]

This relationship links the stress-based approach (in terms of the stress intensity factor) to the energetic one (by Griffith and associated to the concept of the energy release rate).

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**Energy-based criterion**

\( G < G_c \) \quad \text{no crack propagation}  \\
\( G = G_c \) \quad \text{crack propagation}  \\
\( G_c = \frac{1 - \nu^2}{E} K_{Ic}^2 \) \quad \text{critical energy release rate}

**Stress-based criterion**

\( K_I < K_{Ic} \) \quad \text{no crack propagation}  \\
\( K_I = K_{Ic} \) \quad \text{crack propagation}  \\
\( K_{Ic} = \text{critical stress intensity factor} \)
Irwin theory

Generalized Irwin equation

\[ G(c, A) = \frac{1}{E'} \left( K_I^2 + K_{II}^2 \right) + \frac{K_{III}^2}{2\mu} \]

\[ E' = \begin{cases} \frac{E}{1 - \nu^2} & \text{plane stress (thin plates)} \\ \frac{E}{(1 - \nu^2)} & \text{plane strain (thick plates)} \end{cases} \]

Energy-based criterion

\[ G < G_c \quad \text{no crack propagation} \]
\[ G = G_c \quad \text{crack propagation} \]

\[ G_c = \left[ \frac{1}{E'} \left( K_I^2 + K_{II}^2 \right) + \frac{K_{III}^2}{2\mu} \right]_c = \frac{K_c^2}{E'} \quad \text{critical energy release rate} \]

Remark

G: thermodynamic force responsible of crack propagation.
No pointwise criterion but energetic (at a higher scale than the point one) approach

Stress-based criterion

\[ K_I^2 + K_{II}^2 + \frac{E'}{2\mu} K_{III}^2 < K_c^2 \quad \text{no crack propagation} \]
\[ K_I^2 + K_{II}^2 + \frac{E'}{2\mu} K_{III}^2 = K_c^2 \quad \text{crack propagation} \]

\[ K_c = \left[ K_I^2 + K_{II}^2 + \frac{E'}{2\mu} K_{III}^2 \right]_c \quad \text{critical equivalent stress intensity factor} \]