Composite materials comprising bimodular domains

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Meccanica dei Materiali e della Frattura
Complementi di Scienza delle Costruzioni
Bimodular Materials

$E^+$ : tension Young’s modulus
$E^-$ : compression Young’s modulus
Bimodular Materials


\[ W(\mathbf{E}) \begin{cases} \text{continuous} \\ \text{wholewise continuously differentiable} \\ \text{piecewise twice continuously differentiable} \end{cases} \]

\[ \Sigma(\mathbf{E}) \begin{cases} \text{continuous} \\ \text{piecewise continuously differentiable} \end{cases} \]

\[ \mathcal{C}(\mathbf{E}) \text{ piecewise continuous} \]

**Strain space**

\[ J \coloneqq \{ \mathbf{E} \in E \mid g(\mathbf{E}) = 0 \} \]

\[ E_- \coloneqq \{ \mathbf{E} \in E \mid g(\mathbf{E}) < 0 \} \]

\[ E_+ \coloneqq \{ \mathbf{E} \in E \mid g(\mathbf{E}) > 0 \} \]

\[ E \coloneqq J \cup E_- \cup E_+ \]

\[ g(\mathbf{E}) = \mathbf{N} \cdot \mathbf{E} \]

**Graphs and Equations**

\[ W(\mathbf{E}) \]

- Continuous
- Wholewise continuously differentiable
- Piecewise twice continuously differentiable

\[ \frac{d}{d\mathbf{E}} \]

\[ \Sigma(\mathbf{E}) \]

- Continuous
- Piecewise continuously differentiable

\[ \frac{d}{d\mathbf{E}} \]

\[ \mathcal{C}(\mathbf{E}) \]

Piecewise continuous
Bimodular Materials
Continuity conditions


\[
W(E) := \begin{cases} 
W_-(E) & \text{if } g(E) \leq 0 \\
W_+(E) & \text{if } g(E) \geq 0 
\end{cases}
\]

\[
\Sigma(E) := \begin{cases} 
\Sigma_-(E) = \nabla_E W_-(E) & \text{if } g(E) \leq 0 \\
\Sigma_+(E) = \nabla_E W_+(E) & \text{if } g(E) \geq 0 
\end{cases}
\]

\[
\mathbb{C}(E) := \begin{cases} 
\mathbb{C}_-(E) = \nabla^2_E W_-(E) & \text{if } g(E) < 0 \\
\mathbb{C}_+(E) = \nabla^2_E W_+(E) & \text{if } g(E) > 0 
\end{cases}
\]
Bimodular Materials

Continuity conditions


\[
W(\mathbf{E}) := \begin{cases} W_-(\mathbf{E}) & \text{if } g(\mathbf{E}) \leq 0 \\ W_+(\mathbf{E}) & \text{if } g(\mathbf{E}) \geq 0 \end{cases}
\]

\[
W(\mathbf{E}) = W_{-}(\mathbf{E}) = W_{+}(\mathbf{E}) \quad \forall \mathbf{E} \mid g(\mathbf{E}) = 0
\]

\[
\Sigma(\mathbf{E}) := \begin{cases} \Sigma_{-}(\mathbf{E}) = \nabla_{\mathbf{E}} W_{-}(\mathbf{E}) & \text{if } g(\mathbf{E}) \leq 0 \\ \Sigma_{+}(\mathbf{E}) = \nabla_{\mathbf{E}} W_{+}(\mathbf{E}) & \text{if } g(\mathbf{E}) \geq 0 \end{cases}
\]

\[
\Sigma(\mathbf{E}) = \Sigma_{-}(\mathbf{E}) = \Sigma_{+}(\mathbf{E}) \quad \forall \mathbf{E} \mid g(\mathbf{E}) = 0
\]

\[
\mathcal{C}(\mathbf{E}) := \begin{cases} \mathcal{C}_{-}(\mathbf{E}) = \nabla_{\mathbf{E}}^{2} W_{-}(\mathbf{E}) & \text{if } g(\mathbf{E}) < 0 \\ \mathcal{C}_{+}(\mathbf{E}) = \nabla_{\mathbf{E}}^{2} W_{+}(\mathbf{E}) & \text{if } g(\mathbf{E}) > 0 \end{cases}
\]

\[
[\mathcal{C}(\mathbf{E})] := \mathcal{C}_{+}(\mathbf{E}) - \mathcal{C}_{-}(\mathbf{E}) = s(\mathbf{E}) \nabla_{\mathbf{E}} g(\mathbf{E}) \otimes \nabla_{\mathbf{E}} g(\mathbf{E}) \quad \forall \mathbf{E} \mid g(\mathbf{E}) = 0
\]
\[ C(E) = \begin{cases} \mathbb{C}_-(E) = \nabla_E^2 W_-(E) & \text{if } g(E) < 0 \\ \mathbb{C}_+(E) = \nabla_E^2 W_+(E) & \text{if } g(E) > 0 \end{cases} \quad g(E) = \mathbf{N} \cdot \mathbf{E} \]

\[ C(E) := \begin{cases} 3K_+ J + 2\mu_+ K & \text{if } \mathbf{N} \cdot \mathbf{E} < 0 \\ 3K_- J + 2\mu_- K & \text{if } \mathbf{N} \cdot \mathbf{E} > 0 \end{cases} \]

\[ [C(E)] := C_+ - C_- = s(E) \nabla_E g(E) \otimes \nabla_E g(E) \quad \forall E \mid g(E) = 0 \]

\[ [C(E)] := C_+ - C_- = s(E) \mathbf{N} \otimes \mathbf{N} \quad \forall E \mid \mathbf{N} \cdot \mathbf{E} = 0 \]

\[ [C(E)] := C_+ - C_- := 3(K_+ - K_-) J + 2(\mu_+ - \mu_-) K \]

\[ J = \frac{1}{3} I \otimes I \quad K = I - J \]
\[ C(E) := \begin{cases} C_-(E) = V_E^2 W_-(E) & \text{if } g(E) < 0 \\ C_+(E) = V_E^2 W_+(E) & \text{if } g(E) > 0 \end{cases} \]

\[ g(E) = \mathbf{N} \cdot \mathbf{E} \]

\[ [C(E)] := C_+ - C_- = s(E) \mathbf{V}_E g(E) \otimes \mathbf{V}_E g(E) \quad \forall \mathbf{E} \mid g(E) = 0 \]

\[ [C(E)] := C_+ - C_- = s(E) \mathbf{N} \otimes \mathbf{N} \quad \forall \mathbf{E} \mid \mathbf{N} \cdot \mathbf{E} = 0 \]

\[ [C(E)] := 3(\mathbf{J}_+ - \mathbf{J}_-) \mathbf{J} + 2(\mu_+ - \mu_-) \mathbf{K} \]

\[ \mathbf{J} = \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \quad \mathbf{K} = \mathbf{I} - \mathbf{J} \]

\[ \mu_- = \mu_+ = \mu \]

\[ [C(E)] := 3(\mathbf{K}_+ - \mathbf{K}_-) \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \]

\[ s = \mathbf{K}_+ - \mathbf{K}_- \quad \mathbf{N} = \mathbf{I} \Rightarrow \quad g(E) = \mathbf{N} \cdot \mathbf{E} = \mathbf{I} \cdot \mathbf{E} = \text{tr}(\mathbf{E}) \]

Bimodular Materials
Isotropic symmetry


\[
\mathcal{C}(\mathbf{E}) := \begin{cases} 
\mathcal{C}_- (\mathbf{E}) = V_E^2 W_-(\mathbf{E}) & \text{if } g(\mathbf{E}) < 0 \\
\mathcal{C}_+ (\mathbf{E}) = V_E^2 W_+(\mathbf{E}) & \text{if } g(\mathbf{E}) > 0
\end{cases}
\]

\[
[\mathcal{C}(\mathbf{E})] := \mathcal{C}_+ (\mathbf{E}) - \mathcal{C}_- (\mathbf{E}) = s(\mathbf{E}) V_E g(\mathbf{E}) \otimes V_E g(\mathbf{E}) \\
\forall \mathbf{E} \mid g(\mathbf{E}) = 0
\]

\[
[\mathcal{C}(\mathbf{E})] := \mathcal{C}_+ - \mathcal{C}_- = s \mathbf{N} \otimes \mathbf{N} \\
\forall \mathbf{E} \mid \mathbf{N} \cdot \mathbf{E} = 0
\]

\[
[\mathcal{C}(\mathbf{E})] := 3(K_+ - K_-) | \quad \mu_- = \mu_+ = \mu
\]

\[
s = K_+ - K_- \quad \mathbf{N} = \mathbb{I} \Rightarrow \quad g(\mathbf{E}) = \mathbb{I} \cdot \mathbf{E} = \text{tr}(\mathbf{E})
\]

\[
g(\mathbf{E}) = 0 \quad \rightarrow \quad \text{tr}(\mathbf{E}) = 0
\]

\[
\mathcal{C}(\mathbf{E}) := \begin{cases} 
3K_- \mathbb{I} + 2\mu_- \mathbb{K} & \text{if } \text{tr}(\mathbf{E}) < 0 \\
3K_+ \mathbb{I} + 2\mu_+ \mathbb{K} & \text{if } \text{tr}(\mathbf{E}) > 0
\end{cases}
\]
Aim

What is the macroscopic mechanical response of composites comprised of Curnier bimodular materials?

How

Analytical procedure
Aim

What is the macroscopic mechanical response of composites comprised of Curnier bimodular materials?

How

Analytical procedure
Aim

What is the macroscopic mechanical response of composites comprised of Curnier bimodular materials?

How

Computational procedure
Matrix $\rightarrow$ isotropic *piecewise*-linearly-elastic

\[
K(\varepsilon) = \begin{cases} 
K_- = 2.42 \text{ GPa} & \text{if } \text{tr}(\varepsilon) < 0 \\
K_+ = 20.12 \text{ GPa} & \text{if } \text{tr}(\varepsilon) > 0
\end{cases}
\]

$\mu = 1.89 \text{ GPa}$

Inclusion $\rightarrow$ isotropic linearly-elastic

$K_i = 189 \text{ GPa}$

$\mu_i = 179 \text{ GPa}$
Computational procedure

Homogeneous strain boundary conditions

Definition of the local elastic moduli of each material

Uniaxial

Equi-biaxial

Purely hydrostatic

Purely deviatoric

Uniaxial: $\mathbf{E} = \begin{bmatrix} \varepsilon_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Equi-biaxial: $\mathbf{E} = \begin{bmatrix} \varepsilon_0 & 0 & 0 \\ 0 & \varepsilon_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Purely hydrostatic: $\mathbf{E} = \begin{bmatrix} \varepsilon_0 & 0 & 0 \\ 0 & \varepsilon_0 & 0 \\ 0 & 0 & \varepsilon_0 \end{bmatrix}$

Purely deviatoric: $\mathbf{E} = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Computational procedure

10-node tetrahedral elements
# elements > $10^5$

Homogeneous strain boundary conditions

Definition of the local elastic moduli of each material

Updating of $K(x, y, z)$

FEM analysis

Results

$K(x, y, z) = \begin{cases} K_- & \text{if } \text{tr}(\varepsilon(x, y, z)) < 0 \\ K_+ & \text{if } \text{tr}(\varepsilon(x, y, z)) > 0 \end{cases}$
Computational procedure

10-node tetrahedral elements
# elements > 10^5

Homogeneous strain boundary conditions

Definition of the local elastic moduli of each material

Updating of $K(x, y, z)$

FEM analysis

To Do List

1. 2. 3. 4. 5.

MATLAB

$K(x, y, z) = \begin{cases} 
K_- & \text{if } \text{tr}(\varepsilon(x, y, z)) < 0 \\
K_+ & \text{if } \text{tr}(\varepsilon(x, y, z)) > 0 
\end{cases}$
Homogenization approaches
Nonlinear composites

The question of predicting the effective constitutive relations (or equivalently the effective potential) for nonlinear phases is considered in the literature from different standpoints.

The simplest methods are extensions or modifications of the secant method and of the incremental method. These schemes proceed in three successive steps:
(a) First, the constitutive relations of each individual phase is linearized in an appropriate manner. This is done pointwisely and leads to consider the nonlinear composite as a linear composite with infinitely many phases with local elastic moduli varying from point to point.
(b) Then, the problem is reduced to that of estimating the effective properties of a linear comparison composite with a finite number of phases. To this aim, an approximation is introduced by assuming that the local moduli are piecewise uniform. In most cases the regions where the moduli are uniform are precisely the domains occupied by the material phases.
(c) Finally the effective linear properties of the linear comparison composite are estimated by a scheme which is relevant for the type of microstructure under consideration. These linear effective properties are used to define the nonlinear effective properties of the actual nonlinear composite.

Intraphase heterogeneity

\[ \mathbb{C}^{hom}(\varepsilon(\mathbf{x})) \]

\[ \mathbb{C}^{hom}(\bar{\varepsilon}_m, \bar{\varepsilon}_{eq}) \]

Secant method

Modified Secant method

\[ \mathbb{C}^{hom}(\bar{\varepsilon}_m, \bar{\varepsilon}_{eq}) \]

\( \alpha = \# \text{ phase} \)

First order moments:

\[ \bar{\varepsilon}_m = tr(\varepsilon)^\alpha \quad \text{and} \quad \bar{\varepsilon}_{eq} = \bar{\varepsilon}_{eq}^\alpha \]

Second order moments:

\[ \bar{\varepsilon}_m^\alpha = \sqrt{tr(\varepsilon)^2}^\alpha \quad \text{and} \quad \bar{\varepsilon}_{eq}^\alpha = \sqrt{\varepsilon_{eq}^2}^\alpha \]

Fluctuations:

\[ F_m = \sqrt{tr(\varepsilon)^2}^\alpha - (tr(\bar{\varepsilon}^\alpha))^2 \]

\[ F_{eq} = \sqrt{\varepsilon_{eq}^2}^\alpha - (\varepsilon_{eq} \bar{\varepsilon}^\alpha)^2 \]

Intraphase heterogeneity

Macroscopic stress and strain

\[ \Sigma = \Sigma_m \mathbb{I} + \Sigma_{\text{dev}} \]

\[ \Sigma_m = \frac{\text{tr}(\Sigma)}{3} \quad \Sigma_{eq} = \sqrt{\frac{3 \Sigma_{\text{dev}} : \Sigma_{\text{dev}}}{2}} \]

\[ E = E_m \mathbb{I} + E_{\text{dev}} \]

\[ E_m = \frac{\text{tr}(E)}{3} \quad E_{eq} = \sqrt{\frac{E_{\text{dev}} : E_{\text{dev}}}{2}} \]

First and second order moments of the local strain field

\[ \bar{\varepsilon}_m^\alpha = \text{tr}(\varepsilon)^\alpha \quad \bar{\varepsilon}_{eq}^\alpha = \bar{\varepsilon}_{eq}^\alpha \]

\[ \bar{\varepsilon}_m^\alpha = \sqrt{\text{tr}(\varepsilon)^2}^\alpha \quad \bar{\varepsilon}_{eq}^\alpha = \sqrt{\varepsilon_{eq}^2}^\alpha \]

Fluctuations

\[ F_m = \sqrt{\text{tr}(\varepsilon)^2 - (\text{tr}(\bar{\varepsilon}^\alpha))^2} \quad F_{eq} = \sqrt{(\varepsilon_{eq})^2 - (\varepsilon_{eq}(\bar{\varepsilon}^\alpha))^2} \]
the same behavior is observed in the three considered study cases
the macroscopic response is **bilinear**
Results
Equi-biaxial loading

- $E_m = 0.04 \quad E_{eq} = 0.035$

- $E_m = -0.04 \quad E_{eq} = 0.035$

- Even though the strain field does vary at the local length scale, those fluctuations do not result in any intraphase heterogeneity.
Results
Equi-biaxial loading

First and second order moments

- A jump is observed at \( \text{tr}(E) = 0 \) whereas, away from zero values of the macroscopic strain, fluctuations are constants.
Intraphase heterogeneity

Equi-biaxial loading ($E_m > 0$)

$\varepsilon_m(x, y, z)$

$K(x, y, z)$

- $K_i$
- $K_+$
- $K_-$

*intraphase heterogeneity*

*geometric symmetry*
Equi-biaxial loading ($E_m > 0$)

Purely deviatoric loading ($E_{eq} = 0.02$)

Results

Intraphase heterogeneity

Geometric symmetry

\[
\varepsilon_m(x, y, z) = K_i + K_+ (x, y, z) + K_-(x, y, z)
\]

\[
\varepsilon_m(x, y, z) = K_i + K_+ (x, y, z) + K_-(x, y, z)
\]
Results

Intraphase heterogeneity

Equi-biaxial loading ($E_m > 0$)

- Purely deviatoric loading ($E_{eq} = 0.02$)

- intraphase heterogeneity
- geometric symmetry

$\varepsilon_m(x, y, z)$

$K(x, y, z)$

- $K_i$
- $K_+$
- $K_-$

- intraphase heterogeneity
- geometric symmetry
Due to the occurrence of a strongly localized intraphase heterogeneity, the macroscopic hydrostatic stress assumes non-null values even if the material is challenged with purely deviatoric boundary conditions.
The extension of the previously-observed intraphase heterogeneity reduces as the imposed hydrostatic strain increases.
• The macroscopic response becomes nonlinear in the vicinity of null values of hydrostatic strain.
Conclusions

- The macroscopic response of composites comprised of bimodular materials has been numerically investigated.

- Computations have been performed also with the aim to support the development of nonlinear homogenization theoretical models, based - for instance - on the classical and modified secant moduli approaches. The obtained results, presented in terms of the strain moments and fluctuations, furnish useful benchmarking evidence against with those models might be potentially assessed.

- Results suggest that the bimodularity leads to the development of strongly-localized intraphase heterogeneity, thus resulting in a complex constitutive response of the material at the macroscopic length scale.
E. Monaldo, A. Lucchetta, S. Brach, D. Kondo, G. Vairo. Homogenization of composite materials comprising bimodular phases, 10th European Solid Mechanics Conference (ESMC 2018), July 2018, Bologna, Italy.

Francesco Chirianni
«Risposta meccanica di materiali compositi a fasi bimodulari: un approccio di omogeneizzazione computazionale»

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thank you